ON TRANSVERSE VIBRATIONS OF BARS

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1. The differential equations of motion for a vibrating bar, taking into account effects of rotary inertia and shear forces on the deflection, have the following form [1]:

$$\frac{1}{c_1^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2} + Cc_2^2\left(\frac{\partial u}{\partial x} - \psi\right), \qquad C\frac{\partial^2 u}{\partial t^2} = Cc_2^2\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial\psi}{\partial x}\right) + q \qquad (1)$$

Here ψ is the slope of the deflected axis of the bar without effects of shear, $u = u(x, t)$ the transverse displacement of the bar, and $q = q(x, t)$ the external force. Moreover,

$$C = \frac{\rho A}{EI}, \qquad c_1^2 = \frac{E}{\rho}, \qquad c_2^2 = \frac{kG}{\rho}$$

where A is the area of the cross-section of the bars; ρ is the volume density; k a factor depending on the shape of the cross-section; EI the flexural rigidity; G the shear modulus; c_1 and c_2 are the velocities of wave propagation during the transverse vibrations [2]. To obtain a unique determination of the vibrational process we must also specify initial and boundary conditions. Let us assume for simplicity that at $t = 0$

$$\psi = \frac{\partial\psi}{\partial t} = 0, \qquad u = \frac{\partial u}{\partial t} = 0 \qquad (2)$$

The boundary conditions have one of the following forms [3]:

$$u = \frac{\partial\psi}{\partial x} = 0, \qquad u = \psi = 0, \qquad \frac{\partial\psi}{\partial x} = \frac{\partial u}{\partial x} - \psi = 0 \qquad (3)$$

Elimination of ψ from (3) is possible only in the case of a simply supported bar, i.e. for the first boundary condition in (3), or for an infinite bar [3]. In these cases we obtain the following equation for the displacement $u = u(x, t)$

$$\frac{\partial^4 u}{\partial x^4} - \left(\frac{1}{c_1^2} + \frac{1}{c_2^2}\right)\frac{\partial^4 u}{\partial x^2\,\partial t^2} + \frac{1}{c_1^2c_2^2}\frac{\partial^4 u}{\partial t^4} + c\frac{\partial^2 u}{\partial t^2} = \frac{1}{EICc_1^2c_2^2}\left(\frac{\partial^2}{\partial t^2} - c_1^2\frac{\partial^2}{\partial x^2} + Cc_1^2c_2^2\right)q \qquad (4)$$

with the initial conditions at $t = 0$

$$u = \frac{\partial u}{\partial t} = 0, \qquad \frac{\partial^2 u}{\partial t^2} = \frac{1}{EIC}q, \qquad \frac{\partial^3 u}{\partial t^3} = \frac{1}{EIC}\frac{\partial q}{\partial t} \tag{5}$$

2. As an example of application of equation (4), consider vibrations of an inifinite bar acted upon by an arbitrary external force q. The Laplace transform of relationship (4) with initial conditions (5) has the following form:

$$L(U) = \frac{d^4U}{dx^4} - p^2\left(\frac{1}{c_1^2} + \frac{1}{c_2^2}\right)\frac{d^2U}{dx^2} + p^2\left(\frac{p^2}{c_1^2c_2^2} + C\right)U = \\ = \frac{1}{EICc_1^2c_2^2}\left(p^2 - c_1^2\frac{d^2}{dx^2} + Cc_1^2c_2^2\right)Q \tag{6}$$

where U and Q are Laplace transforms of the displacement and force respectively.

The source function of the operator $L(U)$ is

$$R(x, \xi, p) = \frac{1}{2(n_2^2 - n_1^2)}\left(\frac{1}{n_1}e^{-n_1|x-\xi|} - \frac{1}{n_2}e^{-n_2|x-\xi|}\right) \tag{7}$$

where n_1 and n_2 (Re $n_1 > 0$, Re $n_2 > 0$) are roots of the characteristic equation. Thus, the solution of (6), which tends to zero with $|x| \to \infty$, is represented by

$$U = \frac{1}{EICc_1^2c_2^2}\int_{-\infty}^{\infty} R(x, \xi, p)\left(p^2 - c_1^2\frac{d}{d\xi^2} + Cc_1^2c_2^2\right)Q(\xi, p)\,d\xi$$

Integrating (b) by parts we get

$$U = \frac{1}{2EICc_1^2c_2^2}\int_{-\infty}^{\infty}[\varphi(n_1, n_2)e^{-n_1|x-\xi|} + \varphi(n_2, n_1)e^{-n_2|x-\xi|}]\,Q\,d\xi \tag{8}$$

where

$$\varphi(u, v) = \frac{c_1^2u^2 - Cc_1^2c_2^2 - p^2}{u(u^2 - v^2)} \tag{9}$$

3. Next, we apply formula (8) to investigate the character of elastic deflections of an infinite bar which is vibrating as a result of a constant concentrated force P suddenly applied (at time $t = 0$ and at the point $x = 0$). In this case $q = P\sigma_0(t)\sigma_1(x)$. $\sigma_0(t)$ and $\sigma_1(x)$ are unit step and unit impulse functions respectively.

Substituting into (8), in place of Q, the transforms of the given force, we get

$$\frac{2EI}{P}U = \frac{1}{p(n_2^2 - n_1^2)}\ \frac{1}{n_1}e^{-n_1|x|} - \frac{1}{n_2}e^{-n_2|x|}\Big) + \\ + \frac{1}{Cc_1^2c_2^2p(n_2^2 - n_1^2)}\left(\frac{p^2 - c_1^2n_1^2}{n_1}e^{-n_1|x|} - \frac{p^2 - c_1^2n_2^2}{n_2}e^{-n_2|x|}\right) \tag{10}$$

This equation can be represented in the following form

$$\frac{2EI}{P}U=\frac{1}{p^2Cc_2}\exp\left(-p\frac{|x|}{c_2}\right)+\Phi \tag{11}$$

Applying an inversion formula we obtain,

$$\frac{2EI}{P}u(x,\ t)=\omega(x,\ t)+\varphi(x,\ t) \tag{12}$$

where

$$\omega(x,\ t)=\begin{cases}0 & \text{for } c_2t<|x| \\ \dfrac{c_2t-|x|}{Cc_2^2} & \text{for } c_2t>|x|\end{cases} \tag{13}$$

and $\phi(x,\ t)$ is a function whose derivative with respect to x is zero for $x = 0$.

From the relationships (12) and (13) it follows that, at the point of application of the concentrated force, the slope of the deflected curve of the bar suffers a discontinuity, viz. it has a jump there. This jump is proportional to the magnitude of the applied force and is inversely proportional to the square of the velocity of wave propagation c_2. Indeed, from (12) and (13) we get

$$\left.\frac{\partial u}{\partial x}\right|_{x=+0}-\left.\frac{\partial u}{\partial x}\right|_{x=-0}=-\frac{P}{EICc_2^2} \tag{14}$$

For $c_1,\ c_2\to\infty$ equation (4) is reduced to the classical equation for the transverse vibrations of bars. In this case, as it can be seen from (14), the jump is zero. This fact is well known.

In conclusion we note that if, instead of an infinite bar, we consider a loosely supported bar of finite length $2l$, then for determination of displacements u we obtain the following formula:

$$\frac{2EI}{P}U=\frac{1}{p(n_2^2-n_1^2)}\left[\frac{\sinh n_2(|x|-l)}{n_2\cosh n_2l}-\frac{\sinh n_1(|x|-l)}{n_1\cosh n_1l}\right]+$$
$$+\frac{1}{Cc_1^2c_2^2p(n_2^2-n_1^2)}\left[\frac{p^2-c_1^2n_2^2}{n_2\cosh n_2l}\sinh n_2(|x|-l)-\frac{p^2-c_1^2n_1^2}{n_1\cosh n_1l}\sinh n_1(|x|-l)\right] \tag{15}$$

There is also a jump of the slope of the deflected curve at the point of application of the force P. The magnitude of this jump is

$$-\frac{P}{EICc_2^2}$$

The problem of vibration of an infinite bar under action of a suddenly-applied concentrated force P was considered in [2]. In this work, however, the second terms in (10) and (15) were omitted, and as a result of this, the presence of a jump of the slope of the deflected curve was not detected.

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